# Subdiffusive behavior in a trapping potential: Mean square displacement and velocity autocorrelation function

M. A. Despósito[1,2,*] and A. D. Viñales[1]
[1]Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
[2]Consejo Nacional de Investigaciones Científicas y Técnicas, Buenos Aires, Argentina
(Received 16 February 2009; published 14 August 2009)

A theoretical framework for analyzing stochastic data from single-particle tracking in viscoelastic materials and under the influence of a trapping potential is presented. Starting from a generalized Langevin equation, we found analytical expressions for the two-time dynamics of a particle subjected to a harmonic potential. The mean-square displacement and the velocity autocorrelation function of the diffusing particle are given in terms of the time lag. In particular, we investigate the subdiffusive case. Using a power-law memory kernel, exact expressions for the mean-square displacement and the velocity autocorrelation function are obtained in terms of Mittag-Leffler functions and their derivatives. The behaviors for short-, intermediate-, and long-time lags are investigated in terms of the involved parameters. Finally, the validity of usual approximations is examined.

DOI: 10.1103/PhysRevE.80.021111 PACS number(s): 02.50.−r, 05.40.−a, 05.10.Gg, 05.70.Ln

## I. INTRODUCTION

The viscoelastic properties of complex fluids, such as polymers, colloids, and biological materials, can be derived from the dynamics of individual spherical particles embedded in it [1,2]. Particle tracking microrheology experiments [1–8] are based on the observation of the motion of individual tracer particles. In a typical microrheology experiment, particle positions are recorded in the form of a time sequence and information about the dynamics is essentially extracted by measuring the mean-square displacement (MSD) of the probe particles [9–11].

On the other hand, during the last 2 decades, optical traps (or "optical tweezers") have been increasingly used for position detection, covering a wide range of applications in physics and biology [3,12–17]. In an optical trap, the interaction between the laser and the trapped particle can be approximated by a harmonic potential [18]. If the particle is embedded in a Newtonian fluid, the temporal behavior of its position $X(t)$ is described by the standard Langevin equation

$$m\ddot{X}(t) + \gamma\dot{X}(t') + \kappa X = F(t), \tag{1}$$

where $m$ is the mass of the particle, $\gamma$ is the friction coefficient, $\kappa$ is the trap stiffness, and $F(t)$ represents the random thermal force, which is zero centered and has a flat power spectrum [19]. From an experimental point of view, calibration of the trap stiffness is necessary to determine the trapping force at any position where the trapping potential is harmonic. A usually employed method for calibration in normal viscous fluids is the power spectrum method [19].

Quantitative measurements with optical traps in normal viscous fluids have been applied in numerous microrheology experiments. Optical tweezers microrheological technique was extended to measure the viscoelastic properties of complex fluids [20–22]. However, it has been recently noted that the Langevin equation (1) fails to describe the stochastic motion of the trapped bead when it is embedded in a viscoelastic medium [20,21]. In this case, the dynamics of the particle differs from that in a pure viscous media because the stochastic process exhibits an anomalous diffusive behavior [2,11,22–24]. As a consequence, the use of a trapping complicates the analysis of the obtained data, since the interactions with the viscoelastic environment overlap with the influence of the trapping force [1,22,25]. Moreover, calibration by power spectrum method becomes invalid because it based on the assumption that a bead is trapped in a purely viscous fluid. For example, in the application of optical traps in living cells, it is necessary to take into account the *a priori* unknown viscoelastic properties of the medium to properly calibrate the trap [20].

When particles diffuse through a soft complex fluids or biological materials, the mean-square displacement exhibits a slow relaxation with a power-law decay in the range of large times. In absence of external forces, a theoretical description is now well established. A quantitative analysis of the resulting subdiffusive behavior enables one to extract the rheological properties of the material [1]. Based on a generalized Langevin equation (GLE), Mason and co-workers [5,6] obtained a direct relation in the Laplace domain between the mean-square displacement of free tracer particles and the viscoelastic parameters of the medium.

On the other hand, if the anomalously diffusing particle is subjected to an external harmonic well, the decay of the position autocorrelation function is strongly nonexponential [17,26,27]. For example, the corresponding GLE for a particle in a viscoelastic medium and subjected to a harmonic potential was recently investigated by us in Ref. [27], where analytical expressions for the evolution of mean values and variances in terms of Mittag-Leffler functions were obtained for any temporal range and involved parameters. Nevertheless, from an experimental point of view, it is necessary to get expressions for the two-time correlation functions. For instance, the mean-square displacement can be expressed as

$$\rho(\tau) = \lim_{t\to\infty} \langle [X(t+\tau) - X(t)]^2 \rangle, \tag{2}$$

where $|X(t+\tau)-X(t)|$ is the particle displacement between

*mad@df.uba.ar

 

two time points, $t$ denotes the *absolute time*, while $\tau$ is the so-called *lag time* [28]. Alternative information about the experimentally observed diffusive behavior can be extracted from the normalized velocity autocorrelation function (VACF) [29] defined as [27,30]

$$C_V(\tau) = \lim_{t\to\infty} \frac{\langle V(t+\tau)V(t)\rangle}{\langle V^2(t)\rangle}. \tag{3}$$

Then, to calculate the MSD and the VACF, one must know the behavior of the two-time correlations $\langle X(t+\tau)X(t)\rangle$ and $\langle V(t+\tau)V(t)\rangle$.

The aim of this paper is to obtain analytical expressions for the MSD (2) and VACF (3) for a harmonically bounded particle immersed in a viscoelastic environment. This enables us to provide a theoretical framework to analyze the data obtained by particle tracking microrheology experiments in viscoelastic media and using a trapping potential. For this purpose, in Sec. II, we present the corresponding GLE. The two-time dynamics is obtained, which enables us to calculate the MSD and VACF for arbitrary memory kernels. Section III is devoted to the study of the subdiffusive case. The complete analytical solutions are obtained and approximate expressions for different temporal ranges are given. Moreover, the validity of the overdamped approximation is analyzed. Finally, a summary of our results is presented in Sec. IV.

## II. DIFFUSION IN A HARMONIC WELL

### A. Formal solution for the GLE

In what follows, we consider the dynamics of a spherical bead of mass $m$ immersed in a complex or viscoelastic environment and simultaneously trapped in a harmonic potential. The resulting motion can be described by the following GLE:

$$m\ddot{X}(t) + m\int_0^t dt'\,\gamma(t-t')\dot{X}(t') + m\omega_0^2 X = F(t), \tag{4}$$

where $\omega_0=\sqrt{\kappa/m}$ is the frequency of the trap, $\gamma(t)$ is the dissipative memory kernel representing the viscoelastic friction, and the thermal force $F(t)$ is a zero-centered and stationary random force with a correlation function of the form

$$\langle F(t)F(t')\rangle = C(|t-t'|). \tag{5}$$

The integral term in Eq. (4) represents the dependence of the viscous force on the velocity history and the memory kernel $\gamma(t)$ is related to the noise correlation function $C(t)$ via the second fluctuation-dissipation theorem [31]

$$C(t) = k_B T m\gamma(t), \tag{6}$$

where $T$ is the absolute temperature and $k_B$ is the Boltzmann constant.

Although we consider the one-dimensional GLE (4), our results can be extrapolated to the two- or three-dimensional case, assuming that the resulting motion can be described by two or three uncoupled equations for independent coordinates. This can be achieved if the trap exerts an elastic force of the general form $f=-(\kappa_X X+\kappa_Y Y+\kappa_Z Z)$ and assuming that the local environment surrounding the sphere is isotropic. However, our model could also be used to investigate the case of a trapped particle close to a limiting wall, where the motion becomes anisotropic but the drag term can be separated into components parallel and perpendicular to the surface [32].

The GLE (4) can be formally solved by means of the Laplace transformation. Taking into account the deterministic initial conditions $x_0=X(0)$ and $v_0=\dot{X}(0)$, the evolution of the Laplace transform of the position $X(t)$ reads

$$\hat{X}(s) = x_0\left(\frac{1}{s} - \omega_0^2\hat{I}(s)\right) + \left(v_0 + \frac{1}{m}\hat{F}(s)\right)\hat{G}(s), \tag{7}$$

where $\hat{F}(s)$ is the Laplace transform of the noise. The relaxation function $G(t)$ is the Laplace inversion of

$$\hat{G}(s) = \frac{1}{s^2 + s\hat{\gamma}(s) + \omega_0^2}, \tag{8}$$

where $\hat{\gamma}(s)$ is the Laplace transform of the damping kernel and

$$\hat{I}(s) = \frac{\hat{G}(s)}{s} \tag{9}$$

is the Laplace transform of

$$I(t) = \int_0^t dt'\,G(t'). \tag{10}$$

On the other hand, the Laplace transform of the velocity $V(t)=\dot{X}(t)$ satisfies that

$$\hat{V}(s) = \left(v_0 + \frac{1}{m}\hat{F}(s)\right)\hat{g}(s) - x_0\omega_0^2\hat{G}(s), \tag{11}$$

where

$$\hat{g}(s) = s\hat{G}(s). \tag{12}$$

From Eqs. (7) and (11), a formal expression for the displacement $X(t)$ and the velocity $V(t)$ can be written as

$$X(t) = \langle X(t)\rangle + \frac{1}{m}\int_0^t dt'\,G(t-t')F(t'), \tag{13}$$

$$V(t) = \langle V(t)\rangle + \frac{1}{m}\int_0^t dt'\,g(t-t')F(t'), \tag{14}$$

where

$$\langle X(t)\rangle = x_0[1-\omega_0^2 I(t)] + v_0 G(t), \tag{15}$$

$$\langle V(t)\rangle = v_0 g(t) - x_0\omega_0^2 G(t) \tag{16}$$

are the position and velocity mean values evolution, respectively.

### B. Expressions for the MSD and VACF

To calculate the two-time properties of the dynamical variables involved in the expressions of the MSD (2) and

VACF (3), we will make use of the double Laplace transform technique [33]. Then, from Eqs. (7) and (11), we have

$$\langle \hat{X}(s)\hat{X}(s')\rangle = x_0^2\hat{\chi}(s)\hat{\chi}(s') + v_0^2\hat{G}(s)\hat{G}(s') + x_0v_0[\hat{\chi}(s)\hat{G}(s') + \hat{\chi}(s')\hat{G}(s)] + \frac{1}{m^2}\hat{G}(s)\hat{G}(s')\langle\hat{F}(s)\hat{F}(s')\rangle, \tag{17}$$

$$\langle \hat{V}(s)\hat{V}(s')\rangle = v_0^2\hat{g}(s)\hat{g}(s') + x_0^2\omega_0^4\hat{G}(s)\hat{G}(s') - x_0v_0\omega_0^2[\hat{g}(s)\hat{G}(s') + \hat{g}(s')\hat{G}(s)] + \frac{1}{m^2}\hat{g}(s)\hat{g}(s')\langle\hat{F}(s)\hat{F}(s')\rangle, \tag{18}$$

where

$$\hat{\chi}(s) = \frac{1}{s} - \omega_0^2\hat{I}(s) \tag{19}$$

is the Laplace transform of $\chi(t)=1-\omega_0^2 I(t)$.

In the Appendix, we show how the last term of Eqs. (17) and (18) can be calculated. Inserting expressions (A4) and (A5) into Eqs. (17) and (18) and making a double Laplace inversion, we arrive at

$$\langle X(t)X(t')\rangle = x_0^2\chi(t)\chi(t') + \left(v_0^2 - \frac{k_BT}{m}\right)G(t)G(t') + x_0v_0[\chi(t)G(t') + \chi(t')G(t)] + \frac{k_BT}{m}[I(t) + I(t') - I(|t-t'|)] - \frac{k_BT}{m}\omega_0^2 I(t)I(t'), \tag{20}$$

$$\langle V(t)V(t')\rangle = \frac{k_BT}{m}g(|t-t'|) + \left(v_0^2 - \frac{k_BT}{m}\right)g(t)g(t') + \omega_0^2\left(x_0^2\omega_0^2 - \frac{k_BT}{m}\right)G(t)G(t') - x_0v_0\omega_0^2[g(t)G(t') + g(t')G(t)]. \tag{21}$$

Finally, by considering time lags $\tau > 0$, from Eqs. (20) and (21) we have

$$\langle [X(t+\tau) - X(t)]^2\rangle = \frac{2k_BT}{m}I(\tau) - 2x_0v_0\omega_0^2[G(t+\tau) - G(t)][I(t+\tau) - I(t)] + \left(v_0^2 - \frac{k_BT}{m}\right)[G(t+\tau) - G(t)]^2 + \omega_0^2\left(x_0^2\omega_0^2 - \frac{k_BT}{m}\right)[I(t+\tau) - I(t)]^2, \tag{22}$$

$$\langle V(t+\tau)V(t)\rangle = \frac{k_BT}{m}g(\tau) + \left(v_0^2 - \frac{k_BT}{m}\right)g(t+\tau)g(t) + \omega_0^2\left(x_0^2\omega_0^2 - \frac{k_BT}{m}\right)G(t+\tau)G(t) - x_0v_0\omega_0^2[g(t+\tau)G(t) + g(t)G(t+\tau)]. \tag{23}$$

It is important to notice that the analytical expressions (22) and (23) are valid for all absolute times $t$ and time lags $\tau$. They enable us to obtain the two-time dynamics for an arbitrary memory kernel provided that the fluctuation-dissipation theorem (6) is satisfied. Nevertheless, to evaluate the relevant experimental magnitudes (2) and (3), we must take the limit $t \to \infty$. In this case, these expressions could be simplified as follows. Taking into account the usual assumption that the time-dependent frictional coefficient $\gamma(t)$ goes to zero when $t \to \infty$ [34] and using the final value theorem [35], one gets

$$\lim_{t\to\infty}\gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0. \tag{24}$$

Noticing that the Laplace transform of the relaxation function $I(t)$ defined through Eq. (9) is

$$\hat{I}(s) = \frac{s^{-1}}{s^2 + s\hat{\gamma}(s) + \omega_0^2}, \tag{25}$$

the application of the final value theorem and the use of condition (24) yield [34]

$$I(\infty) = 1/\omega_0^2 \tag{26}$$

and using Eqs. (9) and (12) gives

$$G(\infty) = g(\infty) = 0. \tag{27}$$

Applying these conditions in order to take the limit $t \to \infty$ in Eqs. (22) and (23) and using the definitions (2) and (3), one finally obtains the simpler expressions

$$\rho(\tau) = \frac{2k_BT}{m}I(\tau) \tag{28}$$

and

$$C_V(\tau) = g(\tau). \tag{29}$$

Taking into account Eqs. (26) and (27), the equilibrium value of the MSD is given by

$$\rho(\infty) = \frac{2k_BT}{m\omega_0^2}, \tag{30}$$

while, as expected, the VACF decays to zero, i.e., $C_V(\infty)=0$.

It is worth pointing out that in experimental realizations, the time lag is $\tau_{\min} \le \tau \le \tau_{\max}$, being $\tau_{\min}$ the acquisition time interval and $\tau_{\max}$ the measurement time. Moreover, if $N$ is the number of steps $n$ taken at intervals $\tau_{\min}$, only small values of $n(n < N/10)$ are used. Therefore, it is important to obtain valid expressions for all observational time scales instead of getting only its behavior to large times.

To conclude this section, we will find the extension for a trapped particle of the widely used Mason formula [5,6]. Taking the Laplace transform of Eq. (28) and using the definition (25) of the relaxation function $I(t)$, one gets

$$s\hat{\gamma}(s) = \frac{2k_BT}{m}\frac{1}{s\hat{\rho}(s)} - s^2 - \omega_0^2, \tag{31}$$

which gives a direct relation between the mean-square displacement of the particle and the memory kernel. Finally, expression (31) can be related with the viscoelastic parameters of the complex fluid using the generalized Stokes-Einstein equation [1,5,6]

$$s\hat{\gamma}(s) = 6\pi R s\hat{\eta}(s) = 6\pi R\hat{\mathcal{G}}(s), \tag{32}$$

where $R$ is the radius of the spherical tracer particle, $\hat{\eta}(s)$ is the bulk frequency-dependent viscosity of the medium, and $\hat{\mathcal{G}}(s)$ is the Laplace-transformed shear modulus of the viscoelastic fluid. Equation (32) is based on the implicit assumption that Stokes drag for viscous fluids (no-slip boundary conditions) can be generalized to viscoelastic fluids at all $s$ [6].

## III. SUBDIFFUSIVE BEHAVIOR

Notice that the previous results are valid for any memory kernel that satisfies condition (24). On the other hand, it is well known that in the absence of active transport, the dynamics of the particle in a viscoelastic fluid or complex media is subdiffusive and thus the stochastic process presents a long-time tail noise. The most utilized model to reproduce a subdiffusive behavior is characterized by a noise correlation function exhibiting a power-law time decay [27,36,37]

$$C(t) = C_\lambda\frac{t^{-\lambda}}{\Gamma(1-\lambda)}, \tag{33}$$

where $\Gamma(z)$ is the gamma function [38]. The exponent $\lambda$ for a viscoelastic medium is taken as $0<\lambda<1$. The proportionality coefficient $C_\lambda$ is independent of time but depends on the exponent $\lambda$.

Using the fluctuation-dissipation relation (6), the memory kernel $\gamma(t)$ can be written as

$$\gamma(t) = \frac{\gamma_\lambda}{\Gamma(1-\lambda)}t^{-\lambda}, \tag{34}$$

where $\gamma_\lambda = C_\lambda/k_BT$. Then, its Laplace transform reads

$$\hat{\gamma}(s) = \gamma_\lambda s^{\lambda-1}. \tag{35}$$

In this situation, the Laplace transform of the relaxation function $\hat{I}(s)$ reads

$$\hat{I}(s) = \frac{s^{-1}}{s^2 + \gamma_\lambda s^\lambda + \omega_0^2}. \tag{36}$$

The complete temporal behavior of the relaxation functions $I(t)$, $G(t)$, and $g(t)$ was previously obtained by us in Ref. [27]. Using those results in Eqs. (28) and (29), we have

$$\rho(\tau) = \frac{2k_BT}{m}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\omega_0\tau)^{2k}\tau^2 E^{(k)}_{2-\lambda,3+\lambda k}(-\gamma_\lambda\tau^{2-\lambda}), \tag{37}$$

$$C_V(\tau) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\omega_0\tau)^{2k}E^{(k)}_{2-\lambda,1+\lambda k}(-\gamma_\lambda\tau^{2-\lambda}), \tag{38}$$

where $E_{\alpha,\beta}(y)$ is the generalized Mittag-Leffler function [38] defined by the series expansion

$$E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty}\frac{y^j}{\Gamma(\alpha j+\beta)}, \quad \alpha>0, \quad \beta>0 \tag{39}$$

and $E^{(k)}_{\alpha,\beta}(y)$ is the derivative of the Mittag-Leffler function

$$E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k}E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty}\frac{(j+k)!\,y^j}{j!\,\Gamma(\alpha(j+k)+\beta)}. \tag{40}$$

Using the series expansions (39) and (40), one can realize that the short-time behavior of the MSD reads

$$\rho(\tau) \approx \frac{k_BT}{m}\left\{\tau^2 - \frac{2\gamma_\lambda}{\Gamma(5-\lambda)}\tau^{4-\lambda} - \frac{\omega_0^2}{12}\tau^4\right\}, \tag{41}$$

where the first term shows that the particle undergoes ballistic motion when time is very small [39]. The second term comes from the influence of the viscoelastic medium while the third term corresponds to the fact that the particle begins to "see" the trap. The short-time behavior of the VACF can be obtained in a similar way. In this case, we get

$$C_V(\tau) \approx 1 - \frac{\gamma_\lambda}{\Gamma(3-\lambda)}\tau^{2-\lambda} - \frac{\omega_0^2}{2}\tau^2. \tag{42}$$

On the other hand, for $\gamma_\lambda\tau^{2-\lambda} \gg 1$, the MSD and VACF can be obtained introducing the asymptotic behavior of the Mittag-Leffler function [38]

$$E_{\alpha,\beta}(-y) \sim \frac{1}{y\Gamma(\beta-\alpha)}, \quad y>0 \tag{43}$$

into Eqs. (37) and (38). After some calculations, we have

$$\rho(\tau) \approx \frac{2k_BT}{m\omega_0^2}\left\{1 - E_\lambda\left(-\frac{\omega_0^2}{\gamma_\lambda}\tau^\lambda\right)\right\}, \tag{44}$$

$$C_V(\tau) \approx -\frac{1}{\omega_0^2}\frac{d^2}{d^2\tau}E_\lambda\left(-\frac{\omega_0^2}{\gamma_\lambda}\tau^\lambda\right), \tag{45}$$

where $E_\lambda(y) = E_{\lambda,1}(y)$ denotes the one-parameter Mittag-Leffler function [38].

It is worth pointing out that these expressions can be also obtained discarding the inertial term $s^2$ in Eq. (36). In this case, we get

$$\hat{I}(s) = \frac{s^{-1}}{\gamma_\lambda s^\lambda + \omega_0^2} = \frac{1}{\omega_0^2}\left(\frac{1}{s} - \frac{s^{\lambda-1}}{s^\lambda + \omega_0^2/\gamma_\lambda}\right), \tag{46}$$

and using that the Laplace transform of the Mittag-Leffler function [38]

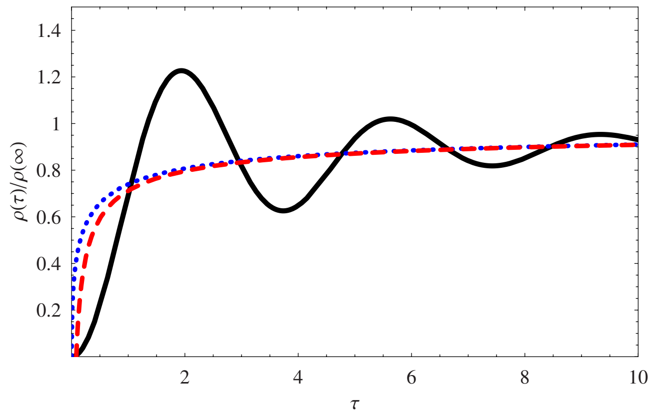


FIG. 1. (Color online) MSD vs time lag for $\lambda=1/2$, $\gamma_\lambda=1$, and $\omega_0=1.4$. Solid line corresponds to the exact solution (37), dotted line to the approximate solution (44), and dashed line to the asymptotic behavior (48).

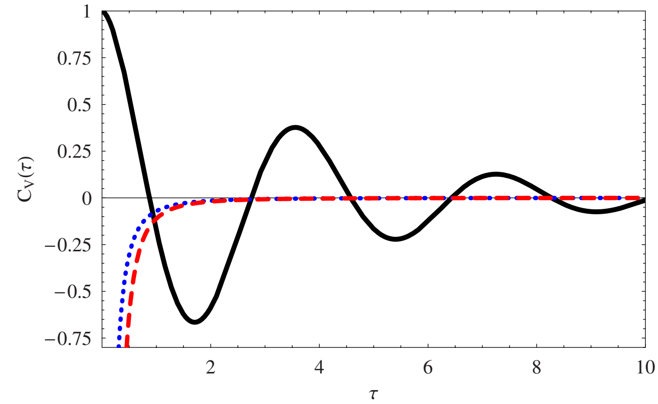


FIG. 3. (Color online) $C_V$ vs time lag for $\lambda=1/2$, $\gamma_\lambda=1$, and $\omega_0=1.4$. Solid line corresponds to the exact solution (38), dotted line to the approximate solution (45), and dashed line to the asymptotic behavior (49).

$$\int_0^\infty e^{-st}E_\alpha(-\gamma t^\alpha)dt=\frac{s^{\alpha-1}}{s^\alpha+\gamma}, \quad (47)$$

one obtains expressions (44) and (45).

Finally, if $\tau^\lambda \gg \gamma_\lambda/\omega_0^2$, the behaviors of the MSD and VACF can be obtained using again the approximation (43). In this case, we get

$$\rho(\tau)\approx\frac{2k_BT}{m\omega_0^2}\left\{1-\frac{\gamma_\lambda}{\omega_0^2}\frac{1}{\Gamma(1-\lambda)}\tau^{-\lambda}\right\}, \quad (48)$$

$$C_V(\tau)\approx-\frac{\gamma_\lambda}{\omega_0^4}\frac{\lambda(\lambda+1)}{\Gamma(1-\lambda)}\tau^{-(\lambda+2)}, \quad (49)$$

showing a pure power-law decay.

In Figs. 1 and 2, we have plotted the MSD vs time lag using the exact solution (37) and the approximations (44) and (48). Note that the exact solution exhibits a nonmonotonic approach to $\rho(\infty)$, while the approximations always present a monotonic behavior.

More pronounced differences occur in the behavior of $C_V(\tau)$ as is evidenced in Figs. 3 and 4. Note that the approximate solutions exhibit a negative velocity autocorrelation for all times, whereas the exact solutions alternate between positive and negative values. This behavior is related to the so-called whip-back effect [27,37].

These behaviors can be understood taking into account that the approximations (44) and (45) only depend on the one-parameter Mittag-Leffler function $E_\lambda(-\omega_0^2\tau^\lambda/\gamma_\lambda)$. It is known that the function $E_\lambda(-t^\lambda)$ is a completely monotone function and tends to zero from above as $t$ tends to infinity for $0<\lambda<1$ [40]. Then, the approximate solutions are always monotonic for every value of $\omega_0$ and $0<\lambda<1$. However, the exact solutions (37) and (38) are expressed as infinite sums of $E^{(k)}_{2-\lambda,\beta}(-\gamma_\lambda\tau^{2-\lambda})$ functions. In this case, the solutions can exhibit a nonmonotonic behavior as is displayed in the previous figures. Then, the overdamped approximation must be used with care in the analysis of the short- and intermediate-time dynamics where the MSD and VACF exhibit a relaxation plus an oscillatory behavior.

Interestingly, Burov and Barkai [41] recently arrived to a similar conclusion using a fractional Langevin equation. Solving the corresponding fractional differential equation in terms of roots of regular polynomials, they analyze the position correlation $\langle X(t)X(0)\rangle$. As in our case, they found that the exact solution can exhibit a nonmonotonic decay, while the overdamped approximation gives a monotonic decay.

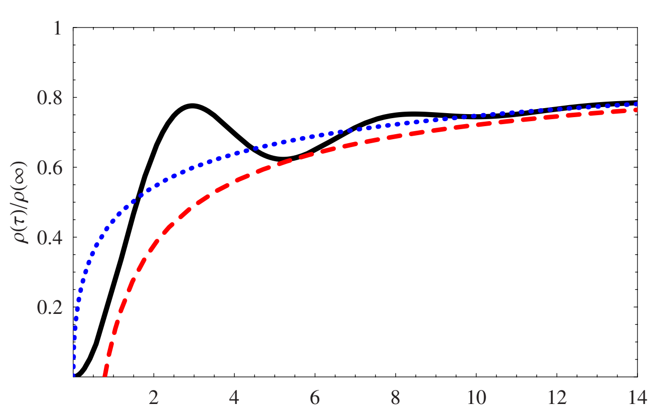


FIG. 2. (Color online) Idem Fig. 1 for $\omega_0=0.8$.

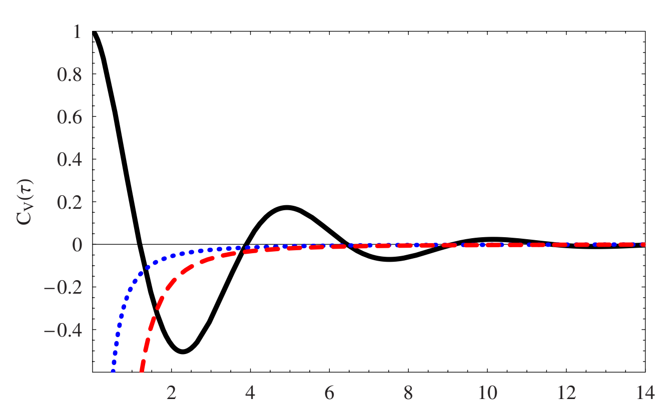


FIG. 4. (Color online) Idem Fig. 3 for $\omega_0=0.8$.

## IV. SUMMARY

In this work, we have obtained the mean-square displacement and the velocity autocorrelation function for a trapped particle and immersed in a complex or viscoelastic media. For this purpose, and starting from a suitable generalized Langevin equation, we have been able to derive analytic expressions for the two-time dynamics of the processes, valid for all absolute times and times lags. We have showed that the MSD and VACF can be expressed as a simple expression when the memory kernel goes to zero for large times. Furthermore, we have presented a generalization of the Mason formula valid when a trapping potential is present.

In particular, we have examined the subdiffusive case, which is a representative example of passive transport in viscoelastic media. Using a power-law memory kernel, exact expressions and valid for all time lags have been obtained in terms of Mittag-Leffler functions and their derivatives. The behavior for short-, intermediate-, and long-time lags are given in terms of the involved parameters. Finally, we have showed that the usual approximations cannot reproduce the nonmonotonic dynamics present in the exact solutions.

In summary, we have presented a theoretical method to account for the effects of the trapping potential in the anomalous behavior of the mean-square displacement and the normalized velocity autocorrelation function of a particle embedded in a complex or viscoelastic environment. We believe that the presented results will be useful to analyze the obtained data from microrheology experiments in viscoelastic media using trapping potentials.

## ACKNOWLEDGMENTS

We would like to thank the anonymous referee for his suggestions, which were very useful to the final form of the paper. This work was performed under Grant No. PICT 31980/05 from Agencia Nacional de Promoción Científica y Tecnológica and Grant No. X099 from Universidad de Buenos Aires, Argentina.

## APPENDIX

To calculate the last term of Eqs. (17) and (18), we make use of a relation given in Ref. [33]. Given any stationary correlation function of the form

$$\langle \Psi(t)\Psi(t')\rangle = A f(|t-t'|), \tag{A1}$$

the corresponding double Laplace transform writes

$$\langle \hat{\Psi}(s)\hat{\Psi}(s')\rangle = A\frac{\hat{f}(s)+\hat{f}(s')}{s+s'}. \tag{A2}$$

Then, the Laplace domain version of the fluctuation-dissipation relation (6) reads [33]

$$\langle \hat{F}(s)\hat{F}(s')\rangle = k_B T m \frac{\hat{\gamma}(s)+\hat{\gamma}(s')}{s+s'}. \tag{A3}$$

After some algebra and using the relations between the kernels $I(t)G(t)$ and $g(t)$, one can find that

$$\hat{G}(s)\hat{G}(s')\langle \hat{F}(s)\hat{F}(s')\rangle = k_B T m\left[\frac{\hat{I}(s)}{s'}+\frac{\hat{I}(s')}{s}-\frac{\hat{I}(s)+\hat{I}(s')}{s+s'}\right] - k_B T m[\hat{G}(s)\hat{G}(s')+\omega_0^2\hat{I}(s)\hat{I}(s')], \tag{A4}$$

$$\hat{g}(s)\hat{g}(s')\langle \hat{F}(s)\hat{F}(s')\rangle = k_B T m\left(\frac{\hat{g}(s)+\hat{g}(s')}{s+s'} - \hat{g}(s)\hat{g}(s') - \omega_0^2\hat{G}(s)\hat{G}(s')\right). \tag{A5}$$


[1] T. A. Waigh, Rep. Prog. Phys. **68**, 685 (2005).
[2] M. T. Valentine, P. D. Kaplan, D. Thota, J. C. Crocker, T. Gisler, R. K. Prud'homme, M. Beck, and D. A. Weitz, Phys. Rev. E **64**, 061506 (2001).
[3] S. Yamada, D. Wirtz, and S. C. Kuo, Biophys. J. **78**, 1736 (2000).
[4] A. W. C. Lau, B. D. Hoffmann, A. Davies, J. C. Crocker, and T. C. Lubensky, Phys. Rev. Lett. **91**, 198101 (2003).
[5] T. G. Mason and D. A. Weitz, Phys. Rev. Lett. **74**, 1250 (1995).
[6] T. G. Mason, K. Ganesan, J. H. van Zanten, D. Wirtz, and S. C. Kuo, Phys. Rev. Lett. **79**, 3282 (1997).
[7] A. J. Levine and T. C. Lubensky, Phys. Rev. Lett. **85**, 1774 (2000).
[8] H. Qian, Biophys. J. **79**, 137 (2000).
[9] Q. Lu and M. J. Solomon, Phys. Rev. E **66**, 061504 (2002).
[10] I. Golding and E. C. Cox, Phys. Rev. Lett. **96**, 098102 (2006).
[11] B. R. Daniels, B. C. Masi, and D. Wirtz, Biophys. J. **90**, 4712 (2006).
[12] K. C. Neuman and S. M. Block, Rev. Sci. Instrum. **75**, 2787 (2004).
[13] B. Lukic, S. Jeney, Ž. Sviben, A. J. Kulik, E.-L. Florin, and L. Forró´, Phys. Rev. E **76**, 011112 (2007).
[14] M. Atakhorrami, J. I. Sulkowska, K. M. Addas, G. H. Koenderink, J. X. Tang, A. J. Levine, F. C. MacKintosh, and C. F. Schmidt, Phys. Rev. E **73**, 061501 (2006).
[15] S. Jeney, B. Lukic, J. A. Kraus, T. Franosch, and L. Forró´, Phys. Rev. Lett. **100**, 240604 (2008).
[16] A. Caspi, R. Granek, and M. Elbaum, Phys. Rev. Lett. **85**, 5655 (2000).
[17] R. Granek and J. Klafter, Europhys. Lett. **56**, 15 (2001).
[18] Y. Deng, J. Bechhoefer, and N. R. Forde, J. Opt. A, Pure Appl. Opt. **9**, S256 (2007).
[19] K. Berg-Sørensen and H. Flyvbjerg, Rev. Sci. Instrum. **75**, 594 (2004); **77**, 063106 (2006).
[20] M. Fischer, K. Berg-Sørensen, and J. Opt, Pure Appl. Opt. **9**, S239 (2007).
[21] G. Pesce, A. C. De Luca, G. Rusciano, P. A. Netti, S. Fusco,

and A. Sasso, J. Opt. A, Pure Appl. Opt. **11**, 034016 (2009).
[22] R. R. Brau *et al.*, J. Opt. A, Pure Appl. Opt. **9**, S103 (2007).
[23] G. Guigas, C. Kalla, and M. Weiss, Biophys. J. **93**, 316 (2007).
[24] I. M. Tolic-Nørrelykke, E.-L. Munteanu, G. Thon, L. Oddershede, and K. Berg-Sørensen, Phys. Rev. Lett. **93**, 078102 (2004).
[25] B. Lukic, S. Jeney, C. Tischer, A. J. Kulik, L. Forró´, and E.-L. Florin, Phys. Rev. Lett. **95**, 160601 (2005).
[26] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[27] A. D. Viñales and M. A. Despósito, Phys. Rev. E **73**, 016111 (2006).
[28] C. Metzner, C. Raupach, D. Paranhos Zitterbart, and B. Fabry, Phys. Rev. E **76**, 021925 (2007).
[29] H. Qian, Biophys. J. **60**, 910 (1991).
[30] J. M. Porra, K. G. Wang, and J. Masoliver, Phys. Rev. E **53**, 5872 (1996).
[31] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, New York, 2001).
[32] T. Franosch and S. Jeney, Phys. Rev. E **79**, 031402 (2009); E. Schäffer, S. F. Nørrelykke, and J. Howard, Langmuir **23**, 3654 (2007); J. Leach, H. Mushfique, S. Keen, R. Di Leonardo, G. Ruocco, J. M. Cooper, and M. J. Padgett, Phys. Rev. E **79**, 026301 (2009).
[33] N. Pottier, Physica A **317**, 371 (2003).
[34] M. A. Despósito and A. D. Viñales, Phys. Rev. E **77**, 031123 (2008).
[35] M. R. Spiguel, *Theory and Problems of Laplace Transform* (McGraw-Hill, New York, 1965).
[36] K. G. Wang, Phys. Rev. A **45**, 833 (1992).
[37] E. Lutz, Europhys. Lett. **54**, 293 (2001).
[38] I. Podlubny, *Fractional Differential Equations* (Academic Press, London, 1999).
[39] K. G. Wang and J. Masoliver, Physica A **231**, 615 (1996).
[40] F. Mainardi and R. Gorenflo, J. Comput. Appl. Math. **118**, 283 (2000).
[41] S. Burov and E. Barkai, Phys. Rev. Lett. **100**, 070601 (2008); Phys. Rev. E **78**, 031112 (2008).